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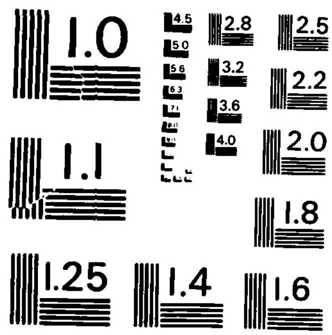

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L^p-STABILITY FOR THE STRONG SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN THE WHOLE SPACE

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SIGNIFICANCE AND EXPLANATION

We consider the motion of a viscous fluid filling the whole space $\mathbf{R}^3$, governed by the classical Navier-Stokes equations (1). Existence of global (in time) regular solutions for that system of non-linear partial differential equations, is still an open problem. From either the mathematical and the physical point of view, an interesting property is the stability (or not) of the (eventual) global regular solutions. Here, we assume that $v_1(t,x)$ is a solution, with initial data $a_1(x)$. For small perturbations of a_1, we want the solution $v_1(t,x)$ being slightly perturbed, too. Due to viscosity, it is even expected that the perturbed solution $v_2(t,x)$ approaches the unperturbed one, as time goes to $+\infty$. This is just the result proved in this paper. To measure the distance between $v_1(t,x)$ and $v_2(t,x)$, at each time t, suitable norms are introduced (L^p-norms).

For fluids filling a bounded vessel, exponential decay of the above distance, is expected. Such a strong result is not reasonable, for fluids filling the entire space. In this case, we prove that the L^p-distance between v_1 and v_2 goes to zero as $(\frac{1}{t})^{3/4}$.

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L^p-STABILITY FOR THE STRONG SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN THE WHOLE SPACE

H. Beirão da Veiga and P. Secchi*

Introduction. Consider the initial value problem for the non-stationary Navier-Stokes equations in the whole space R^3

$$
(1)\qquad
\begin{aligned}
& v' + (v\cdot\nabla)v - \Delta v + \nabla\pi = 0 && \text{in } Q_T \equiv\ ]0,T[\times R^3 ,\\
& \operatorname{div} v = 0 && \text{in } Q_T ,\\
& v_{|t=0} = a && \text{in } R^3 ,\\
& \lim_{|x|\to\infty} v(t,x) = 0 && \text{for } t \in\ ]0,T[,
\end{aligned}
$$

where $T \in\ ]0,\infty]$, $v' = \frac{\partial v}{\partial t}$ and $(v\cdot\nabla)v = \sum_{i=1}^{3} v_i \frac{\partial v}{\partial x_i}$.

The given initial velocity $a(x)$ satisfies $\operatorname{div} a = 0$ in R^3. Moreover, the pressure π is determined by the condition $\lim_{|x|\to\infty} \pi(t,x) = 0$ for $t \in\ ]0,T[$. By a solution of problem (1), we mean a divergence free vector $v(t,x) \in L^q(0,T;L^r)$ for some q, r with $q, r > 2$, such that

$$\int_0^T \int [v\cdot\varphi' + (v\cdot\nabla)\varphi\cdot v + v\cdot\Delta\varphi]dx\, dt = -\int a\, \varphi_{|t=0} dx ,$$

for every regular divergence free vector field $\varphi(t,x)$, with compact support with respect to the space variables and such that $\varphi(T,x) = 0$. We set

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$$L^p = L^p(R^3) , \qquad |\ |_p = \|\ \|_{L^p(R^3)} ,$$

$$N_p(v) = \int_{R^3} |\nabla v|^2 |v|^{p-2} dx .$$

Other notations are standard, or will be introduced in the sequel. Let $p > 3$ and $a_1 \in L^1 \cap L^{p+2}$, $\operatorname{div} a_1 = 0$. Assume that there exists a global solution $v_1 \in L^\infty(0,+\infty;L^{p+2})$ of problem (1), with initial velocity a_1 and pressure π_1. This solution is strong and unique.[1] We prove the following stability result:

Theorem A Assume that the above conditions hold and let $a_2 \in L^1 \cap L^p$, be such that $\operatorname{div} a_2 = 0$. Then, there exists a positive constant y_0 such that, if

(2) $$|a_1 - a_2|_p < y_0 ,$$

there exists a unique solution $v_2 \in C([0,+\infty);L^p)$ of (1) with initial data a_2. This solution verifies the estimate

(3) $$|v_1(t) - v_2(t)|_p \leq C(1 + t)^{-3/4} .$$

The constants y_0 and C depend on p, on the L^1 and L^2 norms of the initial data a_1 and a_2, and on the $L^\infty(0,+\infty;L^{p+2})$ norm of v_1. In particular, by considering initial data a_2 such that

$$|a_2|_1 \leq |a_1|_1 + k_1 , \quad |a_2|_2 \leq |a_1|_2 + k_2 ,$$

where k_1 and k_2 are any positive constants, y_0 and C depend only on k_1, k_2, and on the norms $|a_1|_1$, $|a_1|_2$ and $\|v_1\|_{L^\infty(0,+\infty;L^{p+2})}$ of the unperturbed solution v_1.

---

[1] We refer the reader to the results proved in [2]. See also [5], and references there.

The local existence and uniqueness of a strongly continuous solution $v_2(t)$ with values in $L^2 \cap L^p$ is well known. The bound (3) guarantees the global existence of $v_2(t)$.

The proof of theorem A follows the method introduced in reference [1] in order to study the asymptotic behaviour of the solutions of system (1).

<u>Proof of theorem A</u>. The difference $w = v_2 - v_1$ satisfies the following system:

$$
\text{(4)} \qquad \begin{aligned}
& w' + (v_2 \cdot \nabla)w + (w \cdot \nabla)v_1 - \Delta w + \nabla P = 0 && \text{in } Q_T, \\
& \operatorname{div} w = 0 && \text{in } Q_T, \\
& w|_{t=0} = \alpha && \text{in } R^3,
\end{aligned}
$$

where $P = \pi_2 - \pi_1$ and $\alpha \equiv a_2 - a_1$. Multiply both sides of equation (4) by $|w|^{p-2}w$ and integrate over R^3. After suitable integrations by parts we obtain

$$
\frac{1}{p} \frac{d}{dt} |w|_p^p + N_p(w) + 4 \frac{p-2}{p^2} \int | \nabla |w|^{p/2}|^2 =
$$

$$
- \int (v_2 \cdot \nabla)w \cdot |w|^{p-2}w - \int (w \cdot \nabla)v_1 \cdot |w|^{p-2}w - \int \nabla P \cdot |w|^{p-2}w .
$$

The first term on the right-hand side is zero since v_2 is divergence free. By integrating by parts the other two terms we get

$$
\text{(5)} \qquad \begin{aligned}
& \frac{1}{p} \frac{d}{dt} |w|_p^p + N_p(w) \leq (p - 1) \int |w|^{p-1} |\nabla w| |v_1| + \\
& (p - 2) \int |P| |w|^{p-2} |\nabla w| .
\end{aligned}
$$

Consider the first integral on the right-hand side of (5). By Hölder's and Young's inequalities one has

$$(p-1)\int |w|^{p-1}|\nabla w||v_1| < (p-1)N_p(w)^{1/2}\left(\int |w|^p|v_1|^2\right)^{1/2}$$

(6) $$< \frac{1}{8}N_p(w) + 2(p-1)^2\int |w|^p\,|v_1|^2$$

$$< \frac{1}{8}N_p(w) + 2(p-1)^2|w|^p_{p+2}\,|v_1|^2_{p+2}\ .$$

On the other hand, one can prove the following inequality (see [1], equation (1.14)):

(7) $$|w|^p_{p+2} < C\,N_p(w)^{\frac{3}{p+2}}\,|w|_p^{\frac{p(p-1)}{p+2}}\ ;$$

hence, from (6), one obtains

(8) $$(p-1)\int |w|^{p-1}|\nabla w||v_1| < \frac{1}{4}N_p(w) + C|w|^p_p|v_1|_{p+2}^{\frac{2(p+2)}{p-1}}\ .$$

Consider now the second integral on the right-hand side of (5). By Hölder's and Young's inequalities we have

$$(p-2)\int |P||w|^{p-2}|\nabla w| < (p-2)\left(\int |P|^2|w|^{p-2}\right)^{1/2}N_p(w)^{1/2}$$

(9) $$< 2(p-2)^2\int |P|^2|w|^{p-2} + \frac{1}{8}N_p(w)$$

$$< 2(p-2)^2|P|^2_{\frac{p+2}{2}}|w|^{p-2}_{p+2} + \frac{1}{8}N_p(w)\ .$$

To estimate P, we take the divergence of (4) and obtain

$$-\Delta p = \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}w^i(v_1^{\,j} + v_2^{\,j}) = \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}[w^i(2v_1^{\,j} + w^j)]\ .$$

From the Calderón-Zygmund inequality one has

$$|P|_{\frac{p+2}{2}} < C\sum_{i,j}|w^i(2v_1^j + w^j)|_{\frac{p+2}{2}}\ ;$$

then, by Hölder's inequality,

$$|P|^2_{\frac{p+2}{2}} < C|w|^2_{p+2}(|v_1|^2_{p+2} + |w|^2_{p+2}) .$$

By introducing this last inequality in (9) and by using inequality (7), we get

$$(p-2)\int |P||w|^{p-2}|\nabla w| < C|w|^p_{p+2}(|v_1|^2_{p+2} + |w|^2_{p+2}) + \frac{1}{8} N_p(w)$$

$$(10) \qquad < C\, N_p(w)^{\frac{3}{p+2}} |w|_p^{\frac{p(p-1)}{p+2}} |v_1|^2_{p+2} + C\, N_p(w)^{\frac{3}{p}} |w|_p^{p-1} + \frac{1}{8} N_p(w)$$

$$< \frac{1}{4} N_p(w) + C|w|^p_p |v_1|_{p+2}^{2\frac{p+2}{p-1}} + C\,|w|_p^{\frac{p(p-1)}{p-3}} .$$

Hence, by using inequalities (8) and (10), we get from (5)

$$(11) \qquad \frac{1}{p}\frac{d}{dt}|w|^p_p + \frac{1}{2} N_p(w) < C|w|^p_p |v_1|_{p+2}^{\frac{2(p+2)}{p-1}} + C|w|_p^{\frac{p(p-1)}{p-3}} .$$

On the other hand, one can prove the following Sobolev type inequality (see[1], equation (3.2))

$$N_p(w) > C|w|^p_{3p} ,$$

and, by interpolation, one has

$$|w|_p < |w|_2^{\frac{4}{3p-2}} |w|_{3p}^{\frac{3(p-2)}{3p-2}} .$$

Hence

$$(12) \qquad N_p(w) > C|w|_2^{-\beta}|w|_p^{p+\beta} ,$$

where $\beta = 4p/3(p-2)$. On the other hand, by a result proved in reference [3] (see also [4]) on the L^2-decay of the solutions of the Navier-Stokes equations, one has

(13) $$|w(t)|_2 \le |v_1(t)|_2 + |v_2(t)|_2 \le C\,(t+1)^{-3/4}\,,$$

where C depends only on the L^1 and L^2-norms of the initial velocities. Then, from (12) and (13), we obtain

(14) $$N_p(w) \ge C\,(t+1)^{3\beta/4}|w|_p^{p+\beta}\,.$$

Hence, from (11) and (14), we have

$$\frac{1}{p}\frac{d}{dt}|w|_p^p + C(t+1)^{3\beta/4}|w|_p^{p+\beta} \le C|w|_p^p|v_1|_{p+2}^{\frac{2(p+2)}{p-1}} + C|w|_p^{\frac{p(p-1)}{p-3}}\,,$$

from which we obtain, since v_1 is bounded in L^{p+2}, uniformly in time,

(15) $$\frac{d}{dt}|w|_p + C_1(t+1)^{3\beta/4}|w|_p^{\beta+1} \le C_2|w|_p + C_3|w|_p^{1+\beta+\gamma}\,,$$

where $\gamma = 2p^2/3(p-2)(p-3)$. In (15), C_1 depends on p and on the L^1 and L^2-norms of the initial velocities, C_2 depends on p and $|v_1|_{L^\infty(0,+\infty;L^{p+2})}$, C_3 depends only on p. Consider now the corresponding o.d.e.

(16) $$\begin{aligned} &y'(t) + C_1(t+1)^{3\beta/4}(y(t))^{\beta+1} = C_2 y(t) + C_3(y(t))^{1+\beta+\gamma}\,,\\ &y(0) = |\alpha|_p\,. \end{aligned}$$

We prove now that, if $|\alpha|_p$ is sufficiently small, then $y(t) \le C(t+1)^{-3/4}$. By comparison theorems for o.d.e. it will follow that $|w(t)|_p \le y(t)$; hence (3) holds. Let $t_0 \in \,]0,+\infty[$ be such that

(17) $$t_0 \ge \left(\frac{C_2 + C_3}{C_1}\right)^{4/3\beta} - 1\,.$$

By the continuous dependence on the initial data of the solution of (16), one can find $y_0 > 0$ (depending on p and on C_i, $i = 1,2,3$) sufficiently small such that, if $|\alpha|_p < y_0$, then $y(t) < 1$ for each $t \in [0,t_0]$. Moreover, if there exists $t > t_0$ such that $y(t) = 1$, then from (16) and (17) one has

$$y'(t) = - C_1(t + 1)^{3\beta/4} + C_2 + C_3$$

$$< - C_1(t_0 + 1)^{3\beta/4} + C_2 + C_3 < 0 .$$

This implies $y(t) < 1$, for any $t > 0$. From (16), we then obtain $y(t) < z(t)$ for any $t > 0$, where $z(t)$ is the solution of

$$\begin{aligned} & z'(t) + C_1(t + 1)^{3\beta/4}(z(t))^{\beta+1} = (C_2 + C_3)z(t) , \\ & z(0) = |\alpha|_p . \end{aligned} \tag{18}$$

Equation (18) is of Bernoulli type and its solution is given by

$$z(t) = e^{(C_2+C_3)t} [|\alpha|_p^{-\beta} + \int_0^t e^{\beta(C_2+C_3)s} \beta C_1(s + 1)^{3\beta/4} ds]^{-\frac{1}{\beta}} .$$

Hence

$$z(t)^{\beta}(1 + t)^{3\beta/4} < \frac{(1 + t)^{3\beta/4} e^{\beta(C_2 + C_3)t}}{y_0^{-\infty} + \beta C_1 \int_0^t e^{\beta(C_2 + C_3)s}(s + 1)^{3\beta/4} ds} .$$

By the l'Hôpital theorem one easily shows that the right-hand-side of the above inequality converges to $(C_2 + C_3)/C_1$ as $t \to +\infty$. Since it is equal to y_0^{β} for $t = 0$, it follows that it is bounded, in the interval $[0,+\infty)$ by a constant C (which depends only on C_1, C_2, C_3 and p).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we study L^p-stability, $p > 3$, for strong solutions of the Navier-Stokes system of equations (1), in the whole space $\mathbf{R}^3$. Assume that $v_1 \in L^\infty(0,+\infty; L^{p+2})$ is a solution of (1), with a divergence free initial data $a_1 \in L^1 \cap L^{p+2}$. We prove that there exist two positive constants γ_0 and C such that the following result holds:

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20. ABSTRACT (Continued)

To each divergence free vector field $a_2 \in L^1 \cap L^p$ in the neighborhood $|a_1 - a_2|_p < \gamma_0$, it corresponds a (unique) global solution $v_2 \in C([0,+\infty);L^p)$ of (1), with initial data a_2. Moreover (L^p - asymptotic stability of v_1) one has $|v_1(t) - v_2(t)|_p \leq C(1 + t)^{-3/4}$, $\forall t \in [0,+\infty)$.

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